WITTEN DEFORMATION OF THE ANALYTIC TORSION AND THE SPECTRAL SEQUENCE OF A FILTRATION

MAXIM BRAVERMAN

Abstract. Let F be a flat vector bundle over a compact Riemannian manifold M and let $f : M \to \mathbb{R}$ be a Morse function. Let g^F be a smooth Euclidean metric on F, let $g^F_t = e^{-2tf}g^F$ and let $\rho^{RS}(t)$ be the Ray-Singer analytic torsion of F associated to the metric g^F_t. Assuming that ∇f satisfies the Morse-Smale transversality conditions, we provide an asymptotic expansion for $\log \rho^{RS}(t)$ for $t \to +\infty$ of the form $a_0 + a_1 t + b \log\left(\frac{t}{\pi}\right) + o(1)$, where the coefficient b is a half-integer depending only on the Betti numbers of F. In the case where all the critical values of f are rational, we calculate the coefficients a_0 and a_1 explicitly in terms of the spectral sequence of a filtration associated to the Morse function. These results are obtained as an applications of a theorem by Bismut and Zhang.

0. Introduction

0.1. The *analytic torsion*, ρ^{RS}, introduced by Ray and Singer [RS], is a numerical invariant associated to a flat Euclidean vector bundle F over a compact Riemannian manifold M. It depends smoothly on the Riemannian metric g^{TM} on TM and on the Euclidean metric g^F on F.

Let $f : M \to \mathbb{R}$ be a Morse function. By Witten deformation of the Euclidean bundle F we shall understand the family of metrics

$$g^F_t = e^{-2tf}g^F, \qquad t > 0 \tag{0.1}$$

on F. Let $\rho^{RS}(t)$ be the Ray-Singer torsion of F associated to the metrics g^{TM} and g^F_t.

Burghelea, Friedlander and Kappeler ([BFK3]) have shown (with some additional conditions on f, cf. Section 0.2) that the function $\log \rho^{RS}(t)$ has an asymptotic expansion for $t \to +\infty$ of the form

$$\log \rho^{RS}(t) = \sum_{j=0}^{n+1} a_j t^j + b \log t + o(1). \tag{0.2}$$

The research was supported by grant No. 449/94-1 from the Israel Academy of Sciences and Humanities.

Suppose that the dimension of M is odd and that the Morse function is self-indexing (i.e. $f(x) = index(x)$ for any critical point x of f). Theorem A of [BFK3] implies that in this case (0.2) reduces to the expansion

$$\log \rho^{RS}(t) = a_0 + a_1 t + b \log\left(\frac{t}{\pi}\right) + o(1) \tag{0.3}$$

and also provides formulae for the coefficients a_0, a_1, b.

In the present paper we apply the Bismut-Zhang theorem about the comparison of analytic and combinatorial torsion ([BZ1, Theorem 0.2]) to prove that (0.3) remains true in the general case, when $\dim M$ is not necessarily odd and the Morse function is not necessarily self-indexing.

In the case where all the critical values of f are rational we calculate explicitly the coefficients of the asymptotic expansion (0.3) in terms of the spectral sequence of a filtration associated to f. These calculations are based on the descriptions of the singularities of the torsion obtained by Farber [Fa].

Now we shall discuss the precise formulations of these results.

0.2. Assumptions on f, g^F and g^{TM}. Let $f : M \to \mathbb{R}$ be a Morse function. Denote by ∇f the gradient vector field of f with respect to the metric g^{TM}. Let B be the finite set of zeroes of ∇f.

We shall assume that the following conditions are satisfied (cf. [BFK3, page 5]):

(1) The gradient vector field ∇f satisfies *the Smale transversality conditions* [Sm1, Sm2] (for any two critical points x and y of f the stable manifold $W^s(x)$ and the unstable manifold $W^u(y)$, with respect to ∇f, intersect transversally).

(2) For any $x \in B$, the metric g^F is flat near B and there is a system of coordinates $y = (y^1, \dots, y^n)$ centered at x such that near x

$$g^{TM} = \sum_{i=1}^{n} |dy^i|^2, \quad f(y) = f(x) - \frac{1}{2}\sum_{i=1}^{index(x)} |y^i|^2 + \frac{1}{2}\sum_{i=index(x)+1}^{n} |y^i|^2.$$

0.3. The Milnor torsion. The *Thom-Smale complex* is a finite dimensional complex generated by the fibers of F_x $(x \in B)$ of F whose cohomology is canonically isomorphic to $H^\bullet(M, F)$. The metric g^F on F defines the Euclidean structure on the Thom-Smale complex. The Witten deformation $g_t^F = e^{-2tf} g^F$ of the metric on F determines the deformation of this structure. We shall refer to this deformation as to the Witten deformation of the Thom-Smale complex.

The *Milnor torsion* $\rho^{\mathcal{M}}(t)$ is the torsion of the Thom-Smale complex corresponding to the metric g_t^F on F.

Lemma 0.4. *The function $\rho^{\mathcal{M}}(t)$ admits an asymptotic expansion as $t \to +\infty$ of the form*

$$\log \rho^{\mathcal{M}}(t) = \alpha + \beta t + o(1). \tag{0.4}$$

Lemma 0.4 is proved in Section 2.9.

0.5. The case where all the critical values of f are rational. Suppose now all the critical values $f_1 < \cdots < f_l$ of f are rational. In this situation we shall calculate the coefficients α, β of the asymptotic expansion (0.4).

Assume that $d \in \mathbb{N}$ and $p_1, \dots, p_l \in \mathbb{Z}$ are such that

$$f_i = \frac{p_i}{d}, \qquad (1 \le i \le l). \tag{0.5}$$

After the change of the deformation parameter $t \mapsto \tau = e^{-\frac{t}{d}}$ the Witten deformation of the Thom-Smale complex satisfies the conditions of Theorem 6.6 of [Fa]. Hence, the coefficients α, β of the asymptotic expansion (0.4) may be calculated in terms of the spectral sequence of this deformation (cf. Section 3.2). This spectral sequence admits the following geometric description.

Let $m, k \in \mathbb{Z}$ be such that $(m-k)/d < f(x) < m/d$ for any $x \in M$. We define the filtration $\varnothing = U^0 \imath \dots \imath U^k = M$ on M by

$$U^i = f^{-1}\left[\frac{m-i}{d}, \frac{m}{d}\right] \quad (0 \le i \le k). \tag{0.6}$$

In Section 5 we show that the spectral sequence of the Witten deformation of the Thom-Smale complex my be expressed in terms of the spectral sequence $(E_r^{p,q}, d_r)$ associated with this filtration.

Remark 0.6. The filtration (0.6) and the spectral sequence $(E_r^{p,q}, d_r)$ depend on the choice of d in (0.5). Unfortunately, the spectral sequence associated with seemingly more natural filtration $\varnothing = V^0 \imath \dots \imath V^l = M$, $V^i = f^{-1}[f_i, m/d]$ is not connected to the spectral sequence of the Witten deformation.

Let ρ^{ss} be the torsion of the spectral sequence $(E_r^{p,q}, d_r)$ (cf. Definition 4.14). Note that our definition of the torsion of a spectral sequence is slightly different from [Fa] (cf. Remark 4.15).

Proposition 0.7. *If all the critical values of f are rational, then the coefficients α and β in* (0.4) *are given by the formulae*

$$\alpha = \log \rho^{ss}; \tag{0.7}$$

$$\beta = -\frac{1}{d}\bigg(\sum_{p,q \ge 0} (-1)^{p+q}(p+q) \sum_{r \ge 1} r \Big(\dim E_r^{p,q} - \dim E_{r+1}^{p,q} \Big) \bigg). \tag{0.8}$$

Proposition 0.7 is proved in Section 6.

Remark 0.8. The assumption that the critical values of f are rational does not seem natural. It would be very interesting to obtain formulae for the coefficients α, β of (0.4) without this assumption. These formulae would represent α, β as functions of the critical values of the Morse function f. Unfortunately, those functions are not

continuous. One can show that they can have jumps when the numbers $f_1, \dots, f_l$ are rationally dependent.

Note that, if the critical points of f are not rational, the substitution $t \mapsto \tau = e^{-\frac{t}{d}}$ is not defined and, hence, Farber's theorem can not be applied to the study of the Witten deformation of the Thom-Smale complex.

0.9. Notation. Following [BZ1], we introduce the following definitions.

Let ∇^{TM} be the Levi-Civita connection on TM corresponding to the metric g^{TM}, and let $e(TM, \nabla^{TM})$ be the associated representative of the Euler class of TM in Chern-Weil theory.

Let $\psi(TM, \nabla^{TM})$ be the Mathai-Quillen ([MQ, §7]) $n-1$ current on TM (see also [BGS, Section 3] and [BZ1, Section IIId]) which restriction on $TM/\{0\}$ is induced by a smooth form on the sphere bundle which transgresses the form $e(TM, \nabla^{TM})$.

Let ∇^F denote the flat connection on F and let $\theta(F, g^F)$ be the 1-form on M defined by (cf. [BZ1, Section IVd])

$$\theta(F, g^F) = \mathrm{Tr}\left[(g^F)^{-1}\nabla^F g^F\right]. \tag{0.9}$$

Set

$$\chi(F) = \sum_{i=0}^{n}(-1)^i \dim H^i(M, F), \tag{0.10}$$

$$\chi'(F) = \sum_{i=0}^{n}(-1)^i i \dim H^i(M, F), \tag{0.11}$$

$$\mathrm{Tr}_s^B[f] = \sum_{x\in B}(-1)^{index(x)} f(x). \tag{0.12}$$

Our main result is the following

Theorem 0.10. *Let $\rho^{RS}(t)$ be the analytic torsion corresponding to the metric $g_t^F = e^{-2tf}g^F$.*

(i) The function $\log \rho^{RS}(t)$ admits an asymptotic expansion for $t \to +\infty$ of the form

$$\log \rho^{RS}(t) = a_0 + a_1 t + b \log\left(\frac{t}{\pi}\right) + o(1), \tag{0.13}$$

where

$$b = \frac{n}{4}\chi(F) - \frac{1}{2}\chi'(F) \tag{0.14}$$

and a_0, a_1 are real numbers depending on f, g^F and g^{TM}.

(ii) Assume that all the critical values of f are rational and let the integer d and the spectral sequence $(E_r^{p,q}, d_r)$ be as in Section 0.5*. Then the coefficients a_0 and a_1 in* (0.13) *are given by the formulae*

$$a_0 = \log \rho^{ss} - \frac{1}{2}\int_M \theta(F, g^F)(\nabla f)^* \psi(TM, \nabla^{TM}); \tag{0.15}$$

$$\begin{aligned} a_1 = &- \operatorname{rk}(F) \int_M f e(TM, \nabla^{TM}) \\ &- \frac{1}{d} \sum_{p,q\geq 0} (-1)^{p+q}(p+q) \sum_{r\geq 1} r\Big(\dim E_r^{p,q} - \dim E_{r+1}^{p,q}\Big) + \operatorname{rk}(F) \operatorname{Tr}_s^B[f]. \end{aligned} \tag{0.16}$$

Remark 0.11*.* The Witten deformation is a deformation of an elliptic complex. If this deformation were elliptic with parameter ([Sh], [BFK1]) then, by [BFK1, Theorem A.3], its torsion would have an asymptotic expansion for $t \to \infty$, whose coefficients would be given by local expressions. It fails to be elliptic with parameter precisely at the critical points of the Morse function f. In [BFK3], Burghelea, Friedlander and Kappeler used the Mayer-Vietoris formula for elliptic operators ([BFK1]) to show that its torsion continues to have an asymptotic expansion with coefficients which are not local anymore. Theorem 0.10 provides explicit formulae for these coefficients.

Remark 0.12*.* The connection between the asymptotic behavior of the analytic torsion and the spectral sequence associated with the deformation was discovered by Farber [Fa]. We discuss Farber's results in Section 4.

In [DM], Dai and Melrose have obtained the asymptotic of the Ray-Singer analytic torsion in the adiabatic limit. Their result is also expressed in terms of a spectral sequence.

0.13. The case where the Morse function is self-indexing. Suppose now that the Morse function $f : M \to \mathbb{R}$ is self-indexing and choose $d = 1$ (cf. Section 0.5). For $0 \leq i \leq n$, let m_i denote the number of $x \in B$ of index i. Then (cf. [BZ1, page 30]) the spectral sequence $(E_r^{p,q}, d_r)$ degenerates in the second term and

$$\begin{aligned} \dim E_1^{p,q} &= \begin{cases} m_p \operatorname{rk}(F), & \text{if} \quad q = 0; \\ 0, & \text{if} \quad q \neq 0. \end{cases} \\ \dim E_2^{p,q} &= \begin{cases} \dim H^p(M, F), & \text{if} \quad q = 0; \\ 0, & \text{if} \quad q \neq 0. \end{cases} \end{aligned} \tag{0.17}$$

Hence,

$$\sum_{p,q\geq 0} (-1)^{p+q}(p+q) \sum_{r\geq 1} r\Big(\dim E_r^{p,q} - \dim E_{r+1}^{p,q}\Big) = \operatorname{rk}(F) \operatorname{Tr}_s^B[f] - \chi'(F). \tag{0.18}$$

Also, the torsion ρ^{ss} of the spectral sequence is easily seen to be equal to the Milnor torsion $\rho^{\mathcal{M}}$.

We obtain the following corollary (cf. [BFK3, Theorem A], [Br, Theorem 0.4])

Corollary 0.14. *If the Morse function $f : M \to \mathbb{R}$ is self-indexing, then the coefficients a_0, a_1 of the asymptotic expansion* (0.13) *are given by the formulae*

$$a_0 = \log \rho^{\mathcal{M}} - \frac{1}{2} \int_M \theta(F, g^F)(\nabla f)^* \psi(TM, \nabla^{TM}); \tag{0.19}$$

$$a_1 = -\operatorname{rk}(F) \int_M f e(TM, \nabla^{TM}) + \chi'(F). \tag{0.20}$$

0.15. The method of the proof. Our method is completely different from that of [BFK3]. In [BFK3] the asymptotic expansion is proved by direct analytic arguments and, then is applied to get a new proof of the Ray-Singer conjecture [RS] (which was originally proved by Cheeger [Ch] and Müller [Mü1]).

In Section 8 of the present paper, we use the Bismut-Zhang extension of this conjecture ([BZ1, Theorem 0.2]) to get the following proposition

Proposition 0.16. *As $t \to +\infty$, the following identity holds*

$$\begin{aligned} \log \rho^{RS}(t) - \log \rho^{\mathcal{M}}(t) = \\ -\frac{1}{2} \int_M \theta(F, g^F)(\nabla f)^* \psi(TM, \nabla^{TM}) - t \operatorname{rk}(F) \int_M f e(TM, \nabla^{TM}) \\ + t \operatorname{rk}(F) \operatorname{Tr}_s^B[f] + \left(\frac{n}{4}\chi(F) - \frac{1}{2}\chi'(F)\right) \log\left(\frac{t}{\pi}\right) + o(1). \end{aligned} \tag{0.21}$$

Theorem 0.10 follows from Proposition 0.16, Lemma 0.4 and Proposition 0.7.

Acknowledgments. I am very thankful to Michael Farber for explaining me the role of the spectral sequence of a deformation and for valuable discussions.

1. The torsion of a finite dimensional complex

In this section we follow [BZ1, Section 1a].

1.1. The determinant line. If λ is a real line, let λ^{-1} be the dual line. If E is a finite dimensional vector space, set

$$\det E = \bigwedge^{\max}(E)$$

Let

$$(V^\bullet, \partial) : \ 0 \to V^0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} V^n \to 0 \tag{1.1}$$

be a complex of finite dimensional Euclidean vector spaces. Let $H^\bullet(V)=\bigoplus_{i=0}^n H^i(V)$ be the cohomology of $(V^\bullet,\partial)$. Set

$$\det V^\bullet=\bigotimes_{i=0}^n\Big(\det V^i\Big)^{(-1)^i}, \tag{1.2}$$

$$\det H^\bullet(V)=\bigotimes_{i=0}^n\Big(\det H^i(V)\Big)^{(-1)^i}. \tag{1.3}$$

Then, by [KMu], there is a canonical isomorphism of real lines

$$\det H^\bullet(V)\simeq\det V^\bullet. \tag{1.4}$$

1.2. Two metrics on the determinant line. The Euclidean structure on $V^\bullet$ defines a metric on $\det V^\bullet$. Let $\|\cdot\|_{\det H^\bullet(V)}$ be the metric on the line $\det H^\bullet(V)$ corresponding to this metric via the canonical isomorphism (1.4).

Let ∂^* be the adjoint of ∂ with respect to the Euclidean structure on $C^\bullet(W^u,F)$. Using the finite dimensional Hodge theory, we have the canonical identification

$$H^i(V^\bullet,\partial)\simeq\{v\in V^i:\ \partial v=0,\partial^*v=0\},\quad 0\le i\le n. \tag{1.5}$$

As a vector subspace of V^i, the vector space in the right-hand side of (1.5) inherits the Euclidean metric. We denote by $|\cdot|_{\det H^\bullet(V)}$ the corresponding metric on $\det H^\bullet(V)$. We shall refer to this metric as to the *Hodge metric* on $\det H^\bullet(V)$.

The metrics $\|\cdot\|_{\det H^\bullet(V)}$ and $|\cdot|_{\det H^\bullet(V)}$ do not coincide in general. We shall describe the discrepancy.

1.3. The torsion of a finite dimensional complex. Set $\Delta=\partial\partial^*+\partial^*\partial$ and let $\Pi:V^\bullet\to\operatorname{Ker}\Delta$ be the orthogonal projection. Set $\Pi^\perp=1-\Pi$.

Let N and τ be the operators on $V^\bullet$ acting on V^i $(0\le i\le n)$ by multiplication by i and $(-1)^i$ respectively. If $A\in\operatorname{End}(V^\bullet)$, we define the supertrace $\operatorname{Tr}_s[A]$ by the formula

$$\operatorname{Tr}_s[A]=\operatorname{Tr}[\tau A]. \tag{1.6}$$

For $s\in\mathbb{C}$, set

$$\zeta^V(s)=-\operatorname{Tr}_s\Big[N(\Delta)^{-s}\Pi^\perp\Big].$$

Definition 1.4. The *torsion* of the complex $(V^\bullet,d)$ is the number

$$\rho=\exp\Big(\frac12\frac{d\zeta^V(0)}{ds}\Big). \tag{1.7}$$

We denote by Δ^i $(0 \le i \le n)$ the restriction of Δ on V^i . Let $\{\lambda_j^i\}$ be the set of nonzero eigenvalues of Δ^i. Then

$$\log \rho = \frac{1}{2} \sum_{i,j} (-1)^i i \log \lambda_j^i. \tag{1.8}$$

The following result is proved in [BGS, Proposition 1.5]

$$\| \cdot \|_{\det H^\bullet(V)} = | \cdot |_{\det H^\bullet(V)} \cdot \rho. \tag{1.9}$$

2. The Milnor metric and the Milnor torsion

In this section we recall the definitions of the Milnor metric and the Milnor torsion and prove Lemma 0.4.

2.1. The determinant line of the cohomology. Let $H^\bullet(M,F) = \bigoplus_{i=0}^n H^i(M,F)$ be the cohomology of M with coefficients in F and let $\det H^\bullet(M,F)$ be the line

$$\det H^\bullet(M,F) = \bigotimes_{i=0}^n \Big(\det H^i(M,F)\Big)^{(-1)^i}. \tag{2.1}$$

2.2. The Thom-Smale complex. Let $f : M \to \mathbb{R}$ be a Morse function satisfying the Smale transversality conditions [Sm1, Sm2] (for any two critical points x and y of f the stable manifold $W^s(x)$ and the unstable manifold $W^u(y)$, with respect to ∇f, intersect transversally).

Let B be the set of critical points of f. If $x \in B$, let F_x denote the fiber of F over x and let $[W^u(x)]$ denote the real line generated by $W^u(x)$. For $0 \le i \le n$, set

$$C^i(W^u,F) = \bigoplus_{\substack{x\in B \\ index(x)=i}} [W^u(x)]^* \otimes_{\mathbb{R}} F_x. \tag{2.2}$$

By a basic result of Thom ([Th]) and Smale ([Sm2]) (see also [BZ1, pages 28–30]), there are well defined linear operators

$$\partial : C^i(W^u,F) \to C^{i+1}(W^u,F),$$

such that the pair $(C^\bullet(W^u,F), \partial)$ is a complex and there is a canonical identification of $\mathbb{Z}$-graded vector spaces

$$H^\bullet(C^\bullet(W^u,F),\partial) \simeq H^\bullet(M,F). \tag{2.3}$$

2.3. The Milnor metric. By (1.4) and (2.3), we know that

$$\det H^\bullet(M,F) \simeq \det C^\bullet(W^u,F). \tag{2.4}$$

The metric g^F on F determines the structure of an Euclidean vector space on $C^\bullet(W^u,F)$. This structure induces a metric on $\det C^\bullet(W^u,F)$.

Definition 2.4. The *Milnor metric* $\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}$ on the line $\det H^\bullet(M,F)$ (cf. [BZ1, Section Id]) is the metric corresponding to the above metric on $\det C^\bullet(W^u,F)$ via the canonical isomorphism (2.4).

Remark 2.5. By Milnor [Mi1, Theorem 9.3], if g^F is a flat metric on F, then the Milnor metric coincides with the Reidemeister metric defined through a smooth triangulation of M. In this case $\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}$ does not depend upon f and g^{TM} and, hence, is a topological invariant of the flat Euclidean vector bundle F.

Definition 2.6. The *Hodge-Milnor metric* $|\cdot|^{\mathcal{M}}_{\det H^\bullet(M,F)}$ on $\det H^\bullet(M,F)$ is the metric corresponding to the Hodge metric (cf. Section 1.2) on $\det H^\bullet(C^\bullet(W^u,F),\partial)$ via (2.4).

Definition 2.7. The *Milnor torsion* is the torsion of the Thom-Smale complex (cf. Definition 1.4).

From (1.9), we obtain

$$\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)} = |\cdot|^{\mathcal{M}}_{\det H^\bullet(M,F)} \cdot \rho^{\mathcal{M}}. \tag{2.5}$$

2.8. Deformation of the Milnor metric. The metric $\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}$ depends on the metric g^F. Let $g^F_t = e^{-2tf}g^F$ and let $\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F),t}$ be the corresponding Milnor metric. Recall from Section 0.9 the notation

$$\mathrm{Tr}^B_s[f] = \sum_{x\in B}(-1)^{index(x)}f(x). \tag{2.6}$$

Obviously,

$$\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F),t} = e^{-t\,\mathrm{rk}(F)Tr^B_s[f]} \cdot \|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}. \tag{2.7}$$

2.9. Proof of Lemma 0.4. Assume that g_t is the metric on $C^\bullet(W^u, F)$ induced by the metric $g^F_t = e^{-2tf} g^F$ on F. Let $\mathcal{F} \in \operatorname{End}(C^\bullet(W^u, F))$ which, for $x \in B$, acts on $[W^u(x)]^* \otimes F_x$ by multiplication by $f(x)$. Then, for any $x, y \in C^\bullet(W^u, F)$, we have

$$g_t(x, y) = g_0(e^{-2t\mathcal{F}} x, y). \tag{2.8}$$

Let ∂_t^* be the adjoint of ∂ with respect to the metric g_t. Clearly,

$$\partial_t^* = e^{2t\mathcal{F}} \partial_0^* e^{-2t\mathcal{F}}. \tag{2.9}$$

Set $\Delta_t = \partial\partial_t^* + \partial_t^*\partial$ and denote by Δ_t^i the restriction of Δ_t on $C^i(W^u, F)$. The number $k^i = \dim \operatorname{Ker} \Delta_t^i$ does not depend on t. Hence, the characteristic polynomial $\det(\Delta_t^i - xI)$ of Δ_t^i may be written in the form

$$\det(\Delta_t^i - xI) = x^{k^i} \sum_{j=0}^{\dim V^i - k^i} a_j^i(t) x^j, \tag{2.10}$$

where $a_0^i(t) \neq 0$ is equal to the product of the nonzero eigenvalues of Δ_t^i.

Let $\rho^{\mathcal{M}}(t)$ denote the Milnor torsion corresponding to the metric g_t^F. From (1.8), we see that

$$\log \rho^{\mathcal{M}}(t) = \frac{1}{2} \sum_{i=0}^{n} (-1)^i i a_0^i(t). \tag{2.11}$$

By (2.9), $a_0^i(t)$ is a polynomial in $e^{-2tf_1}, \dots, e^{-2tf_l}$. Hence, for any $0 \le i \le n$, there exist real numbers α^i, β^i such that, for $t \to +\infty$,

$$a_0^i(t) = \alpha^i e^{\beta^i t}(1 + o(1)). \tag{2.12}$$

From (2.11) and (2.12), we obtain

$$\log \rho^{\mathcal{M}}(t) = \frac{1}{2} \sum_{i=0}^{n} (-1)^i i \log \alpha^i + \frac{1}{2} t \sum_{i=0}^{n} (-1)^i i \beta^i + o(1), \tag{2.13}$$

completing the proof of the lemma.

3. The spectral sequence of a deformation

3.1. A deformation of a complex. Let

$$(V^\bullet, \partial_0) : \ 0 \to V^0 \xrightarrow{\partial_0} \cdots \xrightarrow{\partial_0} V^n \to 0 \tag{3.1}$$

be a complex of real vector spaces.

Denote by $\mathcal{O}$ the ring of germs at the origin of real analytic functions of one variable t. By a *deformation of* $(V^\bullet, \partial_0)$ we shall understand a complex

$$(\mathcal{V}^\bullet, \partial) : \ 0 \to \mathcal{V}^0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{V}^n \to 0 \tag{3.2}$$

of $\mathcal{O}$-modules together with a fixed isomorphism between the fiber $\mathcal{V}^\bullet \otimes \mathcal{O}/t\mathcal{O}$ of $(\mathcal{V}^\bullet, \partial)$ at the point $t = 0$ and $(V^\bullet, \partial_0)$.

3.2. The spectral sequence. Given a deformation (3.2) of a complex (3.1), we consider a short exact sequence of complexes

$$0 \to \mathcal{V}^\bullet \xrightarrow{t} \mathcal{V}^\bullet \xrightarrow{q} \mathcal{B}^\bullet \to 0. \tag{3.3}$$

Here $t : \mathcal{V}^\bullet \to \mathcal{V}^\bullet$ is the multiplication by t, $\mathcal{B}^\bullet$ is the quotient complex $\mathcal{B}^\bullet = \mathcal{V}^\bullet / t\mathcal{V}^\bullet$ and q is the quotient map.

The exact sequence (3.3) induces a long exact sequence of cohomology

$$\cdots \to H^m(\mathcal{V}^\bullet) \xrightarrow{t} H^m(\mathcal{V}^\bullet) \xrightarrow{q} H^m(\mathcal{B}^\bullet) \xrightarrow{r} H^{m+1}(\mathcal{V}^\bullet) \to \cdots, \tag{3.4}$$

which may be rewritten as an exact couple

$$\begin{tikzcd} H^\bullet(\mathcal{V}^\bullet) \arrow[rr] & & H^\bullet(\mathcal{V}^\bullet) \arrow[dl] \\ & H^\bullet(\mathcal{B}^\bullet) \arrow[ul] & \end{tikzcd} \tag{3.5}$$

According to the standard rules the latter generates a (Bockstein) spectral sequence $(\mathcal{E}_r^i, d_r)$. To describe it explicitly set

$$\mathcal{Z}_r^i = \left\{ s \in \mathcal{V}^i : \ \partial s \in t^r \mathcal{V}^{i+1} \right\}. \tag{3.6}$$

Then

$$\mathcal{E}_r^i = \begin{cases} V^i \quad \text{for} \quad r = 0, \\ \mathcal{Z}_r^i / (t\mathcal{Z}_{r-1}^i + t^{1-r}\partial \mathcal{Z}_{r-1}^{i-1}) \quad \text{for} \quad r > 0, \end{cases} \tag{3.7}$$

and the differential

$$d_r : \mathcal{E}_r^i \to \mathcal{E}_r^{i+1} \tag{3.8}$$

is the homomorphism induced by the action of $t^{-r}\partial$ on $\mathcal{Z}_r^i$. Then $H^\bullet(\mathcal{E}_r^\bullet, d_r) \simeq \mathcal{E}_{r+1}^\bullet$, i.e. one gets a spectral sequence.

Note that the sequence $(\mathcal{E}_r^i, d_r)$ is completely determined by (3.3).

4. Deformation of the torsion of a finite dimensional complex

In this section we consider a one parameter family (V,∂_t) of finite dimensional Euclidean complexes. This family can be considered as a deformation of a complex and, according to the previous section, gives rise to a spectral sequence. We define the torsion of this spectral sequence. Finally we prove a theorem by Farber (cf. [Fa, Theorem 6.6]) which describes the asymptotic for $t\to 0$ of the torsion $\rho(t)$ of the complex (V,∂_t).

In this section we essentially follow [Fa].

4.1. A family of Euclidean complexes. Let

$$(V^\bullet,\partial_t):\ 0\to V^0 \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_t} V^n\to 0 \tag{4.1}$$

be a one parameter family of complexes of finite dimensional Euclidean vector spaces. We shall assume that the operators

$$\partial_t : V^i \to V^i$$

depend analytically on a parameter t varying within an interval $(-\varepsilon,\varepsilon)$. That means that ∂_t may be represented as a convergent power series

$$\partial_t = \partial_0 + t\partial_1 + \cdots \tag{4.2}$$

with coefficients in $\mathrm{End}\,(V)$.

4.2. The germ complex. Set $\mathcal{V}^i=\mathcal{O}\otimes_{\mathbb{R}} V^i$ $(0\le i\le n)$. The family (4.1) can be understood as a single complex

$$(\mathcal{V}^\bullet,\partial):\ 0\to \mathcal{V}^0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{V}^n\to 0 \tag{4.3}$$

of $\mathcal{O}$-modules, where the differential $\partial:\mathcal{V}^\bullet\to\mathcal{V}^\bullet$ is given by

$$\partial s(t) = \partial_t s(t) \quad (s(t)\in\mathcal{V}). \tag{4.4}$$

The complex (4.3) is called the *germ complex* (cf. [Fa, Section 2.5]).

4.3. The parameterized spectral decomposition. Let ∂_t^* $(-\varepsilon<t<\varepsilon)$ be the adjoint of ∂_t with respect to the Euclidean structure on $V^\bullet$. Set $\Delta_t=\partial_t\partial_t^*+\partial_t^*\partial_t$. For $0\le i\le n$, we denote by Δ_t^i the restriction of Δ_t on V^i.

By a theorem of Rellich [Re, §1, Theorem 1] (see also [Ka, Ch. 7, Theorem 3.9]), there exists a family of analytic curves $\phi_j^i(t)\in\mathcal{V}^i$ $(1\le j\le \dim V^i)$ and a sequence of real valued analytic functions $\lambda_j^i(t)\in\mathcal{O}$ $(1\le j\le\dim V^i)$ such that for any value of t the numbers $\{\lambda_j^i(t)\}$ represent all the repeated eigenvalues of Δ_t^i and $\{\phi_j^i(t)\}$ form a complete orthonormal basis of corresponding eigenvectors of Δ_t^i.

As the operators Δ_t^i are non negative for any t, the functions $\lambda_j^i(t)$ depend only on t^2.

Suppose that $\lambda_j^i(t)$ and $\phi_j^i(t)$ have been numerate so that there exist integers $0 = N_0^i \le N_1^i \le \cdots \le N_{m_i}^i \le N_{m_i+1}^i = \dim V^i$ such that

(1) $\lambda_j^i(t) = t^{2k}\overline{\lambda}_j^i(t)$ with $\overline{\lambda}_j^i(0) \neq 0$ for $N_k^i + 1 \le j \le N_{k+1,}^i$ $0 \le k \le m_i - 1$;
(2) $\lambda_j^i(t) \equiv 0$ for $j \ge N_{m_i}^i + 1$.

4.4. The torsion as a function of the parameter. For each $t \in (-\varepsilon, \varepsilon)$, we shall denote by $\rho(t)$ the torsion of the complex $(V^\bullet, \partial_t)$. The following lemma follows directly from (1.8).

Lemma 4.5. *The function $\rho(t)$ admits an asymptotic expansion for $t \to 0$ of the form*

(4.5)
$$\log \rho(t) = \frac{1}{2}\sum_{i=0}^{n}(-1)^i i \sum_{j=1}^{N_{m_i}^i} \log \overline{\lambda}_j^i(0) + \left(\sum_{i=1}^{n}(-1)^i i \sum_{k=1}^{m_i-1} k(N_{k+1}^i - N_k^i)\right)\log(t) + o(1).$$

Now our goal is to express the right hand side of (4.5) in terms of the spectral sequence of deformation (4.3).

4.6. Let $\mathcal{V}_k^i$ $(0 \le k \le m_i)$ denote the submodule of $\mathcal{V}^i$ generated by the set $\left\{\phi_j^i |\ N_k^i + 1 \le j \le N_{k+1}^i\right\}$. Since the operator ∂_t commutes with Δ_t for any t, we get $\partial \mathcal{V}_k^i \imath \mathcal{V}_k^{i+1}$. Then the equality

$$\langle \Delta_t s(t), s(t)\rangle = \|\partial_t s(t)\|^2 + \|\partial_t^* s(t)\|^2 \tag{4.6}$$

implies the following lemma.

Lemma 4.7. *If $N_k^i + 1 \le j \le N_{k+1}^i$, then*

$$\partial \phi_j^i \in t^k \mathcal{V}_k^{i+1}. \tag{4.7}$$

4.8. The Hodge spectral sequence. Now we shall give a Hodge theoretical description of the spectral sequence $(\mathcal{E}_r^i, d_r)$ associated with the deformation (4.3).

For $r \ge 0$ and $0 \le i \le n$, set

$$H_r^i = \operatorname{span}\left\{\phi_j^i(0) :\ N_r^i + 1 \le j \le \dim V^i\right\} \imath V^i. \tag{4.8}$$

By Lemma 4.7, $\sum_{j=N_r^i+1}^{\dim V^i} a_j^i \phi_j^i(t) \in \mathcal{Z}_r^i$ for any numbers $a_j^i \in \mathbb{R}$. Hence, there is a natural function $\Phi_r : H_r^\bullet \to \mathcal{E}_r^\bullet$ which maps $\sum_{j=N_r^i+1}^{\dim V^i} a_j^i \phi_j^i(0)$ to the image of $\sum_{j=N_r^i+1}^{\dim V^i} a_j^i \phi_j^i(t) \in \mathcal{Z}_r^i$ in $\mathcal{E}_r^i = \mathcal{Z}_r^i / (t\mathcal{Z}_r^i + t^{1-r}\partial \mathcal{Z}_{r-1}^{i-1})$.

Lemma 4.9. *For any $r \ge 0$, the map $\Phi_r : H_r^\bullet \to \mathcal{E}_r^\bullet$ is injective.*

Proof. Suppose that

$$\Phi_r\Big(\sum_{j=N_r^i+1}^{\dim V^i} a_j^i\phi_j^i(0)\Big)=0.$$

Then

$$\sum_{j=N_r^i+1}^{\dim V^i} a_j^i\phi_j^i(t)=t\alpha(t)+t^{1-r}\partial\beta(t),$$

where $\alpha\in\mathcal{Z}_r^i,\ \beta\in\mathcal{Z}_{r-1}^{i-1}$. Hence,

$$\sum_{j=N_r^i+1}^{\dim V^i} a_j^i\phi_j^i(0)=t^{1-r}\partial\beta(t)\Big|_{t=0}. \tag{4.9}$$

By Lemma 4.7,

$$t^{1-r}\partial\beta(t)\Big|_{t=0}\in\operatorname{span}\Big\{\phi_j^i(0):\ N_{r-1}^i+1\le j\le N_r^i\Big\}. \tag{4.10}$$

Hence, (4.9) implies that $\sum_{j=N_r^i+1}^{\dim V^i}a_j^i\phi_j^i(0)=0$. □

By Lemma 4.7, there is a map $\delta_r: H_r^\bullet\to H_r^\bullet$ defined by

$$\delta_r:\sum_{j=N_r^i+1}^{\dim V^i} a_j^i\phi_j^i(0)=t^{-r}\partial\sum_{j=N_r^i+1}^{\dim V^i} a_j^i\phi_j^i(t)\Big|_{t=0}. \tag{4.11}$$

Clearly, $\delta_r^2=0$.

Let $\delta_r^*: H_r^\bullet\to H_r^\bullet$ be the adjoint of δ_r. Using (4.11) and Lemma 4.7, we see that

$$(\delta\delta^*+\delta^*\delta)\phi_j^i(0)=\begin{cases}0, & \text{if}\quad j>N_{r+1}^i;\\ \overline{\lambda}_j^i(0), & \text{if}\quad N_r^i+1\le j\le N_{r+1}^i.\end{cases} \tag{4.12}$$

From (4.8), (4.12), we get

$$H_{r+1}^\bullet=\Big\{h\in H_r^\bullet:\ (\delta\delta^*+\delta^*\delta)h=0\Big\}=\Big\{h\in H_r^\bullet:\ \delta_rh=\delta_r^*h=0\Big\}. \tag{4.13}$$

The following proposition is equivalent to Theorem 3.3 of [KK]. A particular case of this result was proved by Forman ([Fo, Theorem 6]).

Proposition 4.10. *For all $r\ge0$,*

(i) The map $\Phi_r: H_r^\bullet\to\mathcal{E}_r^\bullet$ is an isomorphism.

(ii) $\delta_r=\Phi_r^{-1}d_r\Phi_r$.

Proof. Following [KK], we shall prove the proposition by induction on r.

The case $r = 0$ is obvious. For the inductive step, assume we have proven that $\Phi_r : H_r^\bullet \to \mathcal{E}_r^\bullet$ is an isomorphism and $\delta_r = \Phi_r^{-1} d_r \Phi_r$. Then, by (4.13) and the finite dimensional Hodge theory, $\mathcal{E}_{r+1}^\bullet$ is isomorphic to

$$\Big\{ h \in H_r^\bullet : \ \delta_r h = \delta_r^* h = 0 \Big\} = H_{r+1}^\bullet.$$

Then Lemma 4.9 implies that $\Phi_{r+1} : H_{r+1}^\bullet \to \mathcal{E}_{r+1}^\bullet$ is an isomorphism. From the definition of d_r and δ_r, we obtain $\Phi_{r+1}\delta_{r+1} = d_{r+1}\Phi_{r+1}$, completing the induction step and the theorem. □

Corollary 4.11. *For any* $0 \le i \le n$ *and* $r \ge 0$,

$$\dim \mathcal{E}_r^i = \dim V^i - N_r^i. \tag{4.14}$$

As another corollary we obtain the following result by Farber [Fa, Theorem 1.6]

Proposition 4.12. *The spectral sequence* $(\mathcal{E}_r^\bullet, d_r)$ *stabilizes and the limit term* $\mathcal{E}_\infty^\bullet$ *is isomorphic to the cohomology* $H^\bullet(V^\bullet, \partial_t)$ *for a generic point* t.

4.13. The torsion of the spectral sequence. As a subspace of $V^\bullet$ the vector space $H_r^\bullet$ $(r \ge 0)$ inherits the Euclidean metric. We denote by ρ_r the torsion of the complex $(H_r^\bullet, \delta_r)$ corresponding to this metric. Note that, by Proposition 4.12, $\rho_N = 1$ for sufficiently large N.

Definition 4.14. The *torsion of spectral sequence* $(\mathcal{E}_r^\bullet, d_r)$ is the product

$$\rho^{ss} = \rho_0 \rho_1 \dots \rho_N, \tag{4.15}$$

where N is a sufficiently large number.

Remark 4.15. The torsion of the spectral sequence of a deformation was defined by Farber [Fa, Section 6.5] in slightly different terms. Note that the torsion of a spectral sequence, as it is defined in [Fa], corresponds, in our terms, to the product $\rho_1\rho_2 \dots \rho_N$.

4.16. Farber theorem. Using [Fa, Proposition 6.3], one can easily see that the following theorem is equivalent to [Fa, Theorem 6.6].

Theorem 4.17. *The function* $\rho(t)$ *admits an asymptotic expansion for* $t \to 0$ *of the form*

$$\log \rho(t) = \log \rho^{ss} + \left(\sum_{i=0}^{n} (-1)^i i \sum_{r \ge 1} r \Big(\dim \mathcal{E}_r^i - \dim \mathcal{E}_{r+1}^i \Big) \right) \log(t) + o(1). \tag{4.16}$$

Proof. By (1.8), (4.12), we obtain

$$\log \rho_r = \frac{1}{2}\sum_{i=0}^{n}(-1)^i i \sum_{i=N_r^i+1}^{N_{r+1}^i} \log \overline{\lambda}_j^i(0). \tag{4.17}$$

From (4.5), (4.14), (4.15) and (4.17), we get (4.16). □

5. Deformation of a filtered complex

In this section we apply Theorem 4.17 to a deformation of a filtered complex. The results of this section are closely related to [Fo, Theorem 9].

5.1. The spectral sequence of a filtration. Suppose that

$$(V^\bullet, \partial):\ 0 \to V^0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} V^n \to 0 \tag{5.1}$$

is a filtered complex of real vector spaces with an increasing filtration

$$0 = F_0 V^\bullet \imath F_1 V^\bullet \imath \dots, \qquad V^\bullet = \bigcup_{i \ge 0} F_i V^\bullet. \tag{5.2}$$

Denote by $(E_r^{p,q}, d_r)$ $(r \ge 0)$ the spectral sequence of this filtration and set $E_r^i = \bigoplus_{p+q=i} E_r^{p,q}$. Then (E_r^i, d_r) is also a spectral sequence. We shall need the following description of (E_r^i, d_r) (cf. [BT, §14]).

Let $i : \bigoplus_{p=0}^\infty F_p V^\bullet \to \bigoplus_{p=0}^\infty F_p V^\bullet$ be the map induced by the inclusions $F_p V^\bullet \hookrightarrow F_{p+1} V^\bullet$ $(p \ge 0)$. Consider the exact sequence of complexes

$$0 \to \bigoplus_{p=0}^\infty F_p V^\bullet \xrightarrow{i} \bigoplus_{p=0}^\infty F_p V^\bullet \xrightarrow{j} B^\bullet \to 0. \tag{5.3}$$

The complex $B^\bullet$ is isomorphic to the associated graded complex $grV^\bullet$ of $V^\bullet$.

In a standard way, the exact sequence (5.3) gives rise to a spectral sequence, which is isomorphic to (E_r^i, d_r). It follows, that (E_r^i, d_r) is completely determined by (5.3).

5.2. The Rees complex. We shall construct a complex $(\mathcal{V}^\bullet, \overline{\partial})$ of $\mathcal{O}$-modules as follows

$$\begin{gathered} \mathcal{V}^i = \Big\{ \sum_{m=0}^N v_m t^m :\ v_i \in F_m V^i, N \in \mathbb{N} \Big\}, \qquad \mathcal{V}^\bullet = \bigoplus_{i=0}^n \mathcal{V}^i, \\ \overline{\partial} \sum_{m=0}^N v_p t^p = \sum_{m=0}^N (\partial v_p) t^p. \end{gathered} \tag{5.4}$$

Note that the fiber of $\mathcal{V}^\bullet$ at the point $t = 0$ is isomorphic to the associated graded complex $grV^\bullet$ of $V^\bullet$. Hence, $(\mathcal{V}^\bullet, \overline{\partial})$ is a deformation of $grV^\bullet$ in the sense of Section 3.1.

As in Section 3.2, we construct a short exact sequence of complexes

$$0 \to \mathcal{V}^\bullet \xrightarrow{t} \mathcal{V}^\bullet \xrightarrow{q} \mathcal{B}^\bullet \to 0, \tag{5.5}$$

which induces a spectral sequence $(\mathcal{E}_r^i, \overline{d}_r)$.

Note that, as a real vector space, $\mathcal{V}^\bullet$ is isomorphic to $\bigoplus F_k V^\bullet$ and the multiplication by t corresponds under this isomorphism to the inclusion $i : \bigoplus F_k V^\bullet \hookrightarrow \bigoplus F_k V^\bullet$. Hence, the exact sequences (5.3) and (5.5) are isomorphic. Then so are the spectral sequences (E_r^i, d_r) and $(\mathcal{E}_r^i, \overline{d}_r)$.

5.3. Deformation of the torsion. Suppose that the dimensions of the spaces $V^1, \dots, V^n$ are finite. Then there exists $k \in \mathbb{N}$ such that $F_j V^\bullet = V^\bullet$ for any $j \geq k$. Let $g^{V^0}, \dots, g^{V^n}$ be Euclidean metrics on $V^0, \dots, V^n$. With these assumptions we shall present an equivalent description of deformation (5.4).

Equip $V = \bigoplus_{i=0}^n V^i$ with the metric $g^V = \bigoplus_{i=0}^n g^{V^i}$, which is the orthogonal sum of the metrics $g^{V^0}, \dots, g^{V^n}$.

Let $\Pi_j : V^\bullet \to F_j V^\bullet$ $(0 \leq j \leq k)$ be the orthogonal projection. For $t > 0$, set

$$A_t = \sum_{j=1}^{k} t^j (\Pi_j - \Pi_{j-1}). \tag{5.6}$$

The operator A_t is invertible for any $t \neq 0$.

Define

$$\partial_t = \begin{cases} A_t^{-1} \partial A_t, & \text{for} \quad t \neq 0; \\ \sum_{j=1}^k (\Pi_j - \Pi_{j-1}) \partial (\Pi_j - \Pi_{j-1}), & \text{for} \quad t = 0. \end{cases} \tag{5.7}$$

As in Section 4.1, the family of complexes $(V_\bullet, \partial_t)$ may be considered as a single complex $(\widetilde{V}, \widetilde{\partial})$ of $\mathcal{O}$-modules. Denote by $A : \widetilde{V}^\bullet \to \mathcal{V}^\bullet$ the map defined by the formula

$$A \sum_{i=0}^{N} x_i t^i = \sum_{i=0}^{N} (A x_i) t^i. \tag{5.8}$$

Then A is an isomorphism of complexes of $\mathcal{O}$-modules. Hence, the spectral sequence of deformation $(\widetilde{V}^\bullet, \widetilde{\partial})$ is isomorphic to (E_r^i, d_r). Let ρ^{ss} denote the torsion of this spectral sequence (cf. Definition 4.14). From Theorem 4.17, we get

Proposition 5.4. *Let $\rho(t)$ be the torsion of the complex $(V^\bullet, \partial_t)$ associated to the metric g^V. Then $\rho(t)$ admits an asymptotic expansion for $t \to 0$ of the form*

(5.9)

$$\log \rho(t) = \log \rho^{ss} + \left(\sum_{p,q \geq 0} (-1)^{p+q} (p+q) \sum_{r \geq 1} r \Big(\dim E_r^{p,q} - \dim E_{r+1}^{p,q} \Big) \right) \log(t) + o(1).$$

5.5. Deformation of the metric. Fix $m \in \mathbb{Z}$ and let g_t^V $(t > 0)$ be the metric on V defined by the formula

$$g_t^V(x, y) = g^V(t^{2m} A_t^{-2} x, y). \tag{5.10}$$

We denote by ∂_t^* the adjoint of ∂ with respect to the metric g_t^V. Then $\partial_t^* = A_t^2 \partial^* A_t^{-2}$. Set

$$\Delta_t = \partial \partial_t^* + \partial_t^* \partial. \tag{5.11}$$

Let $\widetilde{\partial}_t^*$ be the adjoint of ∂_t with respect to the metric g^V. Then $\widetilde{\partial}_t^* = A_t \partial^* A_t^{-1}$. Set

$$\widetilde{\Delta}_t = \partial_t \widetilde{\partial}_t^* + \widetilde{\partial}_t^* \partial_t. \tag{5.12}$$

It is easy to see that $\widetilde{\Delta}_t = A_t^{-1} \Delta_t A_t$. Hence, by the definition of the torsion, we obtain the following lemma.

Lemma 5.6. *For any $t > 0$, the torsion of the complex $(V^\bullet, \partial)$ associated to the metric g_t^V is equal to the torsion of the complex $(V^\bullet, \partial_t)$ associated to the metric g^V.*

6. Proof of Proposition 0.7

In this section we assume that all the critical values of f are rational.

6.1. A filtration on the Thom-Smale complex. Recall that the integers d, m and k were defined in Section 0.5. The Thom-Smale complex $(C^\bullet(W^u, F), \partial)$ possesses a natural filtration

$$0 = F_0 C^\bullet(W^u, F) \imath F_1 C^\bullet(W^u, F) \imath \dots \imath F_k C^\bullet(W^u, F) = C^\bullet(W^u, F), \tag{6.1}$$

with

$$F_i C^\bullet(W^u, F) = \bigoplus_{\substack{x \in B \\ f(x) \ge \frac{m-i}{d}}} [W^u(x)]^* \otimes_{\mathbb{R}} F_x \qquad (0 \le i \le k). \tag{6.2}$$

We denote by $(E_r^{p,q}, d_r)$ the spectral sequence of this filtration. This spectral sequence is isomorphic to the spectral sequence of filtration (0.6).

Let ρ^{ss} denote the torsion of the spectral sequence $(E_r^{p,q}, d_r)$.

6.2. Proof of Proposition 0.7. Recall that $\mathcal{F} \in \mathrm{End}\,(C^\bullet(W^u, F))$ was defined in Section 2.9. Let $\Pi_j : C^\bullet(W^u, F) \to F_j(C^\bullet(W^u, F))$ $(0 \leq j \leq k)$ be the orthogonal projection. Set $\tau = e^{-\frac{t}{d}}$. Then $\tau \to 0$ as $t \to +\infty$ and

$$e^{-2t\mathcal{F}} = \tau^{2m}\left(\sum_{j=1}^{k} \tau^j (\Pi_j - \Pi_{j-1})\right)^{-2}. \tag{6.3}$$

Hence, by Proposition 5.4, Lemma 5.6 and (2.8), we see that, for $t \to +\infty$,

(6.4)

$$\log \rho^{\mathcal{M}}(t) = \log \rho^{ss} - \left(\sum_{p,q\geq 0} (-1)^{p+q}(p+q) \sum_{r\geq 1} r\Big(\dim E_r^{p,q} - \dim E_{r+1}^{p,q}\Big)\right)\frac{t}{d} + o(1),$$

which is exactly Proposition 0.7.

7. The Ray-Singer metric and the Ray-Singer torsion

7.1. The L_2 metric on the determinant line. Let $(\Omega^\bullet(M, F), d^F)$ be the de Rham complex of the smooth sections of $\bigwedge(T^*M) \otimes F$ equipped with the coboundary operator d^F. The cohomology of this complex is canonically isomorphic to $H^\bullet(M, F)$.

Let $*$ be the Hodge operator associated to the metric g^{TM}. We equip $\Omega^\bullet(M, F)$ with the inner product

$$\langle \alpha, \alpha' \rangle_{\Omega^\bullet(M,F)} = \int_M \langle \alpha \wedge *\alpha' \rangle_{g^F}. \tag{7.1}$$

By Hodge theory, we can identify $H^\bullet(M, F)$ with the space of harmonic forms in $\Omega^\bullet(M, F)$. This space inherits the Euclidean product (7.1). The L_2 *metric* $|\cdot|^{RS}_{\det H^\bullet(M,F)}$ on $\det H^\bullet(M, F)$ is the metric induced by this product.

7.2. The Ray-Singer torsion. Let d^{F*} be the formal adjoint of d^F with respect to the metrics g^{TM} and g^F.

Set $\Delta = d^F d^{F*} + d^{F*} d^F$ and let $P : \Omega^\bullet(M, F) \to \mathrm{Ker}\,\Delta$ be the orthogonal projection. Set $P^\perp = 1 - P$.

Let N be the operator defining the $\mathbb{Z}$-grading of $\Omega^\bullet(M, F)$, i.e. N acts on $\Omega^i(M, F)$ by multiplication by i.

If an operator $A : \Omega^\bullet(M, F) \to \Omega^\bullet(M, F)$ is trace class, we define its supertrace $\mathrm{Tr}_s[A]$ as in (1.6).

For $s \in \mathbb{C}$, $\mathrm{Re}\; s > n/2$, set

$$\zeta^{RS}(s) = -\,\mathrm{Tr}_s\left[N(\Delta)^{-s} P^\perp\right].$$

By a result of Seeley [Se], $\zeta^{RS}(s)$ extends to a meromorphic function of $s \in \mathbb{C}$, which is holomorphic at $s = 0$.

Definition 7.3. The *Ray-Singer torsion* is the number

$$\rho^{RS} = \exp\left(\frac{1}{2}\frac{d\zeta^{RS}(0)}{ds}\right). \tag{7.2}$$

7.4. The Ray-Singer metric. We now remind the following definition (cf. [BZ1, Definition 2.2]):

Definition 7.5. The *Ray-Singer metric* $\|\cdot\|^{RS}_{\det H^\bullet(M,F)}$ on the line $\det H^\bullet(M,F)$ is the product

$$\|\cdot\|^{RS}_{\det H^\bullet(M,F)} = |\cdot|^{RS}_{\det H^\bullet(M,F)} \cdot \rho^{RS}. \tag{7.3}$$

Remark 7.6. When M is odd dimensional, Ray and Singer [RS, Theorem 2.1] proved that the metric $\|\cdot\|^{RS}_{\det H^\bullet(M,F)}$ is a topological invariant, i.e. does not depend on the metrics g^{TM} or g^F. Bismut and Zhang [BZ1, Theorem 0.1] described explicitly the dependents of $\|\cdot\|^{RS}_{\det H^\bullet(M,F)}$ on g^{TM} and g^F in the case when $\dim M$ is even.

7.7. Bismut-Zhang theorem. Let ∇^{TM} be the Levi-Civita connection on TM corresponding to the metric g^{TM}, and let $e(TM,\nabla^{TM})$ be the associated representative of the Euler class of TM in Chern-Weil theory.

Let $\psi(TM,\nabla^{TM})$ be the Mathai-Quillen ([MQ, §7]) $n-1$ current on TM (see also [BGS, Section3] and [BZ1, Section IIId]).

Let ∇^F be the flat connection on F and let $\theta(F,g^F)$ be the 1-form on M defined by (cf. [BZ1, Section IVd])

$$\theta(F,g^F) = \operatorname{Tr}\Big[(g^F)^{-1}\nabla^F g^F\Big]. \tag{7.4}$$

Now we remind the following theorem by Bismut and Zhang [BZ1, Theorem 0.2].

Theorem 7.8 (Bismut-Zhang). *The following identity holds*

$$\log\left(\frac{\|\cdot\|^{RS}_{\det H^\bullet(M,F)}}{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}}\right)^2 = -\int_M \theta(F,g^F)(\nabla f)^*\psi(TM,\nabla^{TM}). \tag{7.5}$$

7.9. Dependence on the metric. The metrics $\|\cdot\|^{RS}_{\det H^\bullet(M,F)}$ and $\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}$ depend, in general, on the metric g^F. Let $g^F_t = e^{-2tf}g^F$ and let $\|\cdot\|^{RS}_{\det H^\bullet(M,F),t}$ and $\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F),t}$ be the Ray-Singer and Milnor metrics on $\det H^\bullet(M,F)$ associated to the metrics g^F_t and g^{TM}.

By [BZ1, Theorem 6.3]

$$\begin{aligned}\int_M \theta(F,g^F_t)(\nabla f)^*\psi(TM,\nabla^{TM}) = \int_M \theta(F,g^F)(\nabla f)^*\psi(TM,\nabla^{TM})\\ + 2t\operatorname{rk}(F)\int_M fe(TM,\nabla^{TM}) - 2t\operatorname{rk}(F)\operatorname{Tr}^B_s[f].\end{aligned} \tag{7.6}$$

From (7.5) and (7.6), we get

$$\log\left(\frac{\|\cdot\|^{RS}_{\det H^\bullet(M,F),t}}{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F),t}}\right)^2 = -\int_M \theta(F,g^F)(\nabla f)^*\psi(TM,\nabla^{TM}) - 2t\,\mathrm{rk}(F)\int_M fe(TM,\nabla^{TM}) + 2t\,\mathrm{rk}(F)\,\mathrm{Tr}_s^B[f]. \tag{7.7}$$

8. Proof of Proposition 0.16 and Theorem 0.10

8.1. For each $t>0$, we equip $\Omega^\bullet(M,F)$ with the inner product

$$\langle\alpha,\alpha'\rangle_{\Omega^\bullet(M,F),t} = \int_M \langle\alpha\wedge *\alpha'\rangle_{g^F_t}, \tag{8.1}$$

and we denote by $|\cdot|^{RS}_{\det H^\bullet(M,F),t}$ the L_2 metric on $\det H^\bullet(M,F)$ (cf. Section 7.1) associated to this inner product.

Let $|\cdot|^{\mathcal{M}}_{\det H^\bullet(M,F),t}$ be the Hodge-Milnor metric on $\det H^\bullet(M,F)$ (cf. Definition 2.6) associated to the metric g^F_t on F.

From (2.5), (7.3) and (7.7), we get

$$\log\rho^{RS}(t) - \log\rho^{\mathcal{M}}(t) = -\frac{1}{2}\int_M \theta(F,g^F)(\nabla f)^*\psi(TM,\nabla^{TM}) - t\,\mathrm{rk}(F)\int_M fe(TM,\nabla^{TM}) + t\,\mathrm{rk}(F)\,\mathrm{Tr}_s^B[f] + \log\left(\frac{|\cdot|^{\mathcal{M}}_{\det H^\bullet(M,F),t}}{|\cdot|^{RS}_{\det H^\bullet(M,F),t}}\right). \tag{8.2}$$

Proposition 0.16 follows now from (8.2) and the following lemma:

Lemma 8.2. *As $t\to+\infty$,*

$$\log\left(\frac{|\cdot|^{\mathcal{M}}_{\det H^\bullet(M,F),t}}{|\cdot|^{RS}_{\det H^\bullet(M,F),t}}\right) = \left(\frac{n}{4}\chi(F) - \frac{1}{2}\chi'(F)\right)\log\left(\frac{t}{\pi}\right) + o(1). \tag{8.3}$$

Proof. Let d^{F*}_t $(t>0)$ be the formal adjoint of d^F with respect to the inner product (8.1) and let $\Delta_t = d^F d^{F*}_t + d^{F*}_t d^F$.

Let $\Omega^{\bullet,[0,1]}_t(M,F)$ be the direct sum of the eigenspaces of Δ_t associated to eigenvalues $\lambda\in[0,1]$. The pair $(\Omega^{\bullet,[0,1]}_t(M,F),d^F)$ is a subcomplex of $(\Omega^\bullet(M,F),d^F)$ and the inclusion induces an isomorphism of cohomology

$$H^\bullet(\Omega^{\bullet,[0,1]}_t(M,F),d^F) \simeq H^\bullet(M,F). \tag{8.4}$$

We denote by $\|\cdot\|_{\Omega^{\bullet,[0,1]}_t(M,F),t}$ the norm on $\Omega^{\bullet,[0,1]}_t(M,F)$ determined by inner product (8.1) and by $\|\cdot\|_{C^\bullet(W^u,F),t}$ the norm on $C^\bullet(W^u,F)$ determined by g^F_t (cf. Section 2.3).

Recall that $\mathcal{F} \in \mathrm{End}\,(C^\bullet(W^u,F))$ was defined in Section 2.9. Clearly,

$$\|\alpha\|_{C^\bullet(W^u,F),t} = \|e^{-t\mathcal{F}}\alpha\|_{C^\bullet(W^u,F),0}, \qquad (\alpha \in C^\bullet(W^u,F)). \tag{8.5}$$

For an operator $T : C^\bullet(W^u,F) \to \Omega_t^{\bullet,[0,1]}(M,F)$ we denote by T^* its adjoint with respect to the norms $\|\cdot\|_{C^\bullet(W^u,F),0}$ and $\|\cdot\|_{\Omega_t^{\bullet,[0,1]}(M,F),t}$.

In the sequel, $\mathbf{o}(1)$ denotes an element of $\mathrm{End}\,\Big(C^\bullet(W^u,F)\Big)$ which preserves the $\mathbb{Z}$-grading and is $o(1)$ as $t \to \infty$.

By [BZ2, Theorem 6.9], if $t > 0$ is large enough, there exists an isomorphism

$$e_t : C^\bullet(W^u,F) \to \Omega_t^{\bullet,[0,1]}(M,F)$$

of $\mathbb{Z}$-graded Euclidean vector spaces such that

$$e_t^* e_t = 1 + \mathbf{o}(1). \tag{8.6}$$

By [BZ2, Theorem 6.11], for any $t > 0$, there is a quasi-isomorphism of complexes

$$P_t : \Big(\Omega_t^{\bullet,[0,1]}(M,F), d^F\Big) \to \Big(C^\bullet(W^u,F), \partial\Big),$$

which induces the canonical isomorphism

$$H^\bullet(M,F) \simeq H^\bullet(\Omega_t^{\bullet,[0,1]}(M,F), d^F) \simeq H^\bullet(C^\bullet(W^u,F), \partial) \tag{8.7}$$

and such that

$$P_t e_t = e^{t\mathcal{F}} \left(\frac{t}{\pi}\right)^{n/4-N/2} \Big(1 + \mathbf{o}(1)\Big). \tag{8.8}$$

Here $e^{t\mathcal{F}}\left(\frac{t}{\pi}\right)^{n/4-N/2}$ denotes the operator on $C^\bullet(W^u,F)$ which, for $x \in B$, acts on $[W^u(x)]^* \otimes F_x$ by multiplication by $e^{tf(x)}\left(\frac{t}{\pi}\right)^{n/4-index(x)/2}$. In particular, for $t > 0$ large enough, P_t is one to one.

Let P_t^* denote the adjoint of P_t with respect to the norms $\|\cdot\|_{C^\bullet(W^u,F),0}$ and $\|\cdot\|_{\Omega_t^{\bullet,[0,1]}(M,F),t}$. From (8.6), (8.8) we get

$$P_t P_t^* = e^{2t\mathcal{F}} \left(\frac{t}{\pi}\right)^{n/2-N} \Big(1 + \mathbf{o}(1)\Big). \tag{8.9}$$

Denote by $P_t^\#$ the adjoint of P_t with respect to the norms $\|\cdot\|_{C^\bullet(W^u,F),t}$ and $\|\cdot\|_{\Omega_t^{\bullet,[0,1]}(M,F),t}$. Clearly,

$$P_t^\# = P^* \cdot e^{-2t\mathcal{F}}. \tag{8.10}$$

Hence,

$$P_t P_t^\# = \left(\frac{t}{\pi}\right)^{n/2-N} \Big(1 + \mathbf{o}(1)\Big). \tag{8.11}$$

Fix $0 \leq i \leq n$. Let $\sigma \in H^i(M,F)$ and let $\omega_t \in \operatorname{Ker}\Delta_t$ be the harmonic form representing σ.

Denote by $\partial_t^\#$ the adjoint of ∂ with respect to the norms $\|\cdot\|_{C^\bullet(W^u,F),t}$ and $\|\cdot\|_{\Omega_t^{\bullet,[0,1]}(M,F),t}$ and let $\Pi : C^\bullet(W^u,F) \to \operatorname{Ker}(\partial\partial_t^\# + \partial_t^\#\partial)$ be the orthogonal projection. Then $\Pi P_t\omega_t \in C^i(W^u,F)$ corresponds to σ via the canonical isomorphisms (1.5), (8.7).

As P_t commutes with ∂, we see that

$$P_t\omega_t \in \operatorname{Ker}\partial, \qquad \left(\frac{t}{\pi}\right)^{n/2-i}\left(P_t^\#\right)^{-1}\omega_t \in \operatorname{Ker}\partial_t^\#. \tag{8.12}$$

By (8.11), we get $\left(\frac{t}{\pi}\right)^{n/2-i}\left(P_t^\#\right)^{-1}\omega_t = \Big(1+\mathbf{o}(1)\Big)P_t\omega_t$. Then (8.12) implies

$$\Big\|\Pi P_t\omega_t\Big\|_{C^\bullet(W^u,F),t} = \Big\|P_t\omega_t\Big\|_{C^\bullet(W^u,F),t}\Big(1+o(1)\Big). \tag{8.13}$$

It follows from (8.11), (8.13) that

$$\Big\|\Pi P_t\omega_t\Big\|_{C^\bullet(W^u,F),t} = \left(\frac{t}{\pi}\right)^{n/4-i/2}\Big\|\omega_t\Big\|_{\Omega_t^{\bullet,[0,1]}(M,F),t}\Big(1+o(t)\Big). \tag{8.14}$$

By (8.14) and by the definitions of the metrics $|\cdot|^{\mathcal{M}}_{\det H^\bullet(M,F),t}$, $|\cdot|^{RS}_{\det H^\bullet(M,F),t}$, we obtain (8.3). $\square$

The proof of Proposition 0.16 is completed.
Theorem 0.10 follows now from Proposition 0.16, Lemma 0.4 and Proposition 0.7.

References

[BGS] J.-M. Bismut, H. Gillet, C. Soulé, *Analytic torsion and holomorphic determinant bundles, I*, Comm. Math. Phys. **115** (1988) 49-78.

[BZ1] J.-M. Bismut, W. Zhang, *An extension of a theorem by Cheeger and Müller*, Astérisque **205** (1992)

[BZ2] ———, *Milnor and Ray-Singer metrics on the equivariant determinant of a flat vector bundle*, GAFA **4** (1994) 136–212.

[BT] R. Bott, L.W. Tu, *Differential forms in algebraic topology*, Springer-Verlag, 1982.

[Br] M. Braverman, *Witten deformation of Ray-Singer analytic torsion*, Preprint.

[BFK1] D. Burghelea, L. Friedlander, T. Kappeler, *Mayer-Vietoris type formula for determinants of elliptic differential operators*, Journal of Funct. Anal. **107** (1992) 34–65.

[BFK2] ———, *Analytic torsion equals Reidemeister torsion, a new proof*, Preprint (1992), Ohio State University.

[BFK3] ———, *Asymptotic expansion of the Witten deformation of the analytic torsion*, preprint, 1994.

[Ch] J. Cheeger, *Analytic torsion and the heat equation*, Ann. of Math. **109** (1979) 259–300

[DM] X. Dai, R.B. Melrose, *Adiabatic limit of the analytic torsion*, Preprint

[Fa] M. Farber, *Singularities of analytic torsion*, to appear in J. of Diff. Geom.

[Fo] R. Forman, *Hodge theory and spectral sequences*, Topology **33** (1994) 591–611.

[Ka] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, 1966.

[KK] P. Kirk, E. Klassen, *The spectral flow of the odd signature operators and higher Massey products*, Preprint.

[KMu] F.F. Knudsen, D. Mumford, *The projectivity of the moduli spaces of stable curves, I: Preliminaries on "det" and "div"* , Math. Scand. **39** (1976) 19–55.

[MQ] V. Mathai, D. Quillen, *Superconnections, Thom class, and equivariant differential forms*, Topology **25** (1986) 85–110.

[Mi1] J.Milnor, *Whitehead torsion*, Bull. AMS **72** (1966) 358–426

[Mi2] ———, *A duality theorem for Reidemeister torsion*, Ann. of Math. **76** (1962), 137–147.

[Mü1] W. Müller, *Analytic torsion and R-torsion on Riemannian manifolds*, Adv. in Math **28** (1978) 233–305

[Mü2] ———, *Analytic torsion and R-torsion for unimodular representations*, Jour. of AMS **6** (1993) 721–753.

[RS] D. Ray, I. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Adv. in Math **7** (1971) 145–210

[Re] F. Rellich, *Perturbation theory of eigenvalue problems*, Gordon and Breach, 1969.

[Se] R. Seeley, *Complex powers of elliptic operators*, Proc. Symp. Pure and Appl. Math. AMS **10** (1967) 288–307

[Sm1] S. Smale, *On gradient dynamical systems*, Ann. of Math. **74** (1961) 199–206.

[Sm2] ———, *Differentiable dynamical systems*, Bull. Am. Math. Soc. **73** (1967) 747–817.

[Sh] M. Shubin, *Pseudodifferential Operators and Spectral Theory*, Springer Verlag, Berlin, New York, 1980.

[Th] R. Thom, *Sur une partition en cellules associée à une fonction sur une variété*, C.R. Acad. Sc. Paris, **Série A, 228** (1949) 661–692.

[Wi] E. Witten, *Supersymmetry and Morse Theory*, J. of Diff. Geom. **17** (1982) 661–692

School of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv 69978, Israel
E-mail address: maxim@math.tau.ac.il